ON THE VALUATION OF PARIS OPTIONS: FOUNDATIONAL RESULTS

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This paper addresses the valuation of the Paris barrier options of [**CJY**] using the Laplace transform approach. The notion of Paris options is extended such that their valuation is possible at any point during their lifespan. The Laplace transforms of [**CJY**] are modified when necessary, and their basic analytic properties are discussed.

1. Introduction: This paper is the first part of a report on the valuation of the Paris form of barrier options proposed in [**CJY**]. In their standard form European–style barrier options come as puts or calls that are activated or deactivated as soon as their underlying hits a prespecified barrier level. The new idea of the Paris barrier options for cushioning this abruptness is to introduce a systematic delay for these consequences of hitting the barrier to become effective.

To fix ideas consider the Paris down–and–in call option. Suppose the underlying is moving downwards and hits the barrier level. This Paris option is then not immediately activated. Instead, a clock starts ticking at the very moment the underlying passes below the barrier level. It will tick for D consecutive periods in time. Only then, with this delay of D periods the option will be activated if the following two conditions hold. First, one is still during the life span of the option. Second, during any of these last D periods of time the underlying has been staying below the barrier level. To further illustrate this Paris option, suppose D periods before its maturity the option is not yet activated and no clock has started ticking. Then the option will not be activated during its life time. It is already worthless.

This paper addresses foundational questions in valuating Paris options. For instance, a principal difficulty had been pointed out to the author by Pliska. Suppose the clock is already ticking for a Paris option, how should one valuate this option? The necessary extensions of [**CJY**] are discussed in Part I. For valuating Paris options the Laplace transform approach is adopted. Indeed, Laplace transforms of their prices have been computed in [**CJY**], and the discussion of Part II is based on their results. However, pursuing Pliska's suggestions we found it necessary to modify one of these two Laplace transforms. So I am very grateful to Pliska for his remark. In this conncetion I also wish to thank Yor for making available to me corrections to [**CJY**] collected by Jeanblanc–Picqué.

This status quo ante set by [**CJY**] is extended in Part III where a number of insights into the analytical structure of the above Laplace transforms are discussed. More precisely, these Laplace transforms are constructed using a function Ψ whose analytical properties are analyzed. In particular the series of this function is derived and also an asymptotic expansion for it is proved that is uniform on the right–half plane. These results are fundamental for the Laplace inversion of the above Laplace transforms.

Recall that the Laplace transform generally consists of two stages. As a first step, a suit-

ably nice function on the positive real line is transformed into a complex–valued function. This Laplace transform exists on a half–plane sufficiently deep within the right complex half–plane and defines there a holomorphic, i.e., complex analytic, function. As a second step, the Laplace transform has to be inverted to give the desired function on the positive real line. The natural methods in this step are complex analytic. Relevant concepts for the endeavours ahead are recalled in Part 0.

To proceed, recall how for valuating barrier options the relative position of strike price and barrier play a role. For Paris options this list has to be extended and we propose the following notions. If the barrier is equal to or below the strike price we have a standard situation, otherwise we have a perverse situation. Also the relative position of today's price of the underlying to the barrier level matters. We have the first situation if today's price of the underlying is equal to or above the barrier level. If today's price of the underlying is below the barrier level, so that the clock for Paris option has started ticking, we have the second situation. These situations combine into four possible cases. In each of these cases there is a different Laplace transform to be inverted.

With this program in mind the reader now may wish not only to take a look at this paper but also at its companion [**SP**]. The focus of this last paper is on Laplace inversion. It addresses valuating the Paris options in the above first standard case. However, its results about Laplace inversion are typical and fundamental for the remaining three cases.

Part 0   Epitome of methods

2. Certain classes of higher transcendental functions: This paragraph recalls pertinent facts about the the gamma function and the error function from [**L**], [**D**].

Gamma function: The *gamma function* Γ is defined on the right complex half–plane by the improper integral:

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt\,,$$

for any complex number z with positive real part. It interpolates the factorial in the sense that $\Gamma(n+1) = n!$, for any non–negative integer n, and more generally satisfies the *recursion rule* $\Gamma(z+1) = z\Gamma(z)$. The *functional equation* $\Gamma(z)\cdot\Gamma(1-z)\cdot\sin(\pi z) = \pi$ in particular shows how it continues to a meromorphic function on the whole complex plane with simple poles in the non–positive integers as its singularities. It satisfies the *duplication formula* $2^{2z-1}\Gamma(z)\Gamma(z+1/2) = \sqrt{\pi}\,\Gamma(2z)$.

For any complex numbers a, z with positive real part, one has the identity: $\Gamma(z) = a^z \int_0^\infty e^{-at} t^{z-1}\, dt$. Indeed, for a real it is obtained by a change of variable. Applying the identity theorem the resulting identity is then valid on the whole right half–plane.

Unlike the gamma function itself, its reciprocal is holomorphic on the whole complex plane. A repesentation as a contour integral is given by the *Hankel formula*:

$$\frac{x^{a-1}}{\Gamma(a)} = \frac{1}{2\pi i} \int_C e^{xw} w^{-a} \, dw \, ,$$

that is valid for any complex number x with positive real part, and now for any complex number a. Herein C is any of the *Hankel contours* $C_{\varepsilon,R}$ with $0 < \varepsilon < R$, or their limits for ε converging to zero, defined as the following paths of integration: enter from minus infinity
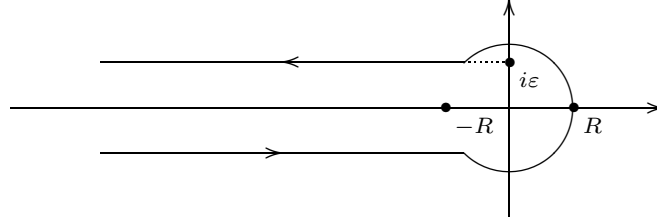

Figure A The Hankel contour $C_{\varepsilon,R}$

on a parallel to the real axis through $-i\varepsilon$ until circle in the orgin with radius R is first hit. Pass counterclockwise on this circle until the parallel to the real axis through $+i\varepsilon$ is hit in the left complex half–plane. Leave on this parallel to minus infinity.

Error function: The *complementary error function* Erfc is the analytic function on the complex plane given by:

$$\text{Erfc}\,(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-w^2} \, dw \, ,$$

for any complex number z, where the path of integration in the last integral starts at z and moves parallel to the real axis with the real part of its elements going to infinity.

3. Laplace transform: This paragraph collects pertinent facts about the Laplace transform from [**D**].

The class of functions considered in the sequel is that of functions of *exponential type*, i.e., of continuous, real–valued functions f on the non–negative real line such that there is a real number a for which $\exp(at)f(t)$ is bounded for any $t > 0$. The *Laplace transform* is the linear operator $\mathbf{L}$ that associates to any function f of exponential type the complex–valued function $\mathbf{L}(f)$, analytic on a suitable complex half–plane, given by:

$$\mathbf{L}(f)(z) = \int_0^\infty e^{-zt} f(t) \, dt \, ,$$

for any complex number z with sufficiently big real part. The *convergence theorem* is that the Laplace transform $\mathbf{L}(f)$ converges on the whole half plane $\{z | \text{Re}\,(z) \geq \text{Re}\,(z_0)\}$ if $\mathbf{L}(f)(z_0)$ exists. So there is a unique $-\infty \leq \sigma = \sigma_f \leq \infty$, the *abscissa of convergence* of $\mathbf{L}(f)$, such that $\mathbf{L}(f)(z)$ converges, for any z with $\text{Re}\,(z) > \sigma$ and does not exist for z with $\text{Re}\,(z) < \sigma$. To fix ideas, consider the function given by $f(t) = \exp(at)$ with a any positive real number so that formally $\mathbf{L}(f)(z) = \int_0^\infty \exp(-(z-a)t) \, dt$. With the absolute

value of $\exp(w)$ being equal to $\exp(\mathrm{Re}(w))$, for any complex number w, this improper integral exists iff the real part of $z-a$ is positive, i.e., iff the real part of z is bigger than a. In this case, $\mathbf{L}(f)(z)$ equals $(z-a)^{-1}$ and is analytic on the half plane $\{z|\mathrm{Re}(z)>a\}$.

That $\mathbf{L}$ is an injection is a consequence of ultimately the Weierstraß approximation theorem. Its inverse, the inverse Laplace transform $\mathbf{L}^{-1}$, is expressed as a contour integral by the *complex inversion formula* of Riemann. For any function H analytic on half–planes $\{\mathrm{Re}(z)\geq x_0\}$ with x_0 any sufficiently big positive real number, it asserts:

$$\mathbf{L}^{-1}(H)(t) = \frac{1}{2\pi i}\int_{x_0-i\infty}^{x_0+i\infty} e^{zt}\cdot H(z)\,dz,$$

for any positive real number t if H satisfies a growth condition at infinity such that the above integral exists. In particular $(\mathbf{L}^{-1}\circ\mathbf{L})(f) = f$ if f is of exponential type.

The convolution of two functions f and g of exponential type is the function of exponential type given by $f*g(t) = \int_0^t f(u)g(t-u)\,du$ for any $t\geq 0$. With du being Lebesgue measure, notice $f*g = g*f$. The *product theorem* for the Laplace transform is that $\mathbf{L}$ transforms convolutions into products:

$$\mathbf{L}(f*g)(z) = \mathbf{L}(f)(z)\cdot\mathbf{L}(g)(z),$$

for any z with $\mathrm{Re}(z)$ sufficiently big. Equivalently, $f*g(t) = \mathbf{L}^{-1}(\mathbf{L}(f)\cdot\mathbf{L}(g))(t)$. The proof is a formal Fubini calculation as soon as one extends functions of exponential type by zero on the non–positive real line, whence $f*g(t)$ equals $\int_0^\infty f(u)g(t-u)\,du$.

There are a number of *shifting principles* for the Laplace transform. First, weighting any function f of exponential type with the function sending any $t>0$ to $\exp(at)$, where a is any real number, introduces a shift by minus a in the Laplace transform:

$$\mathbf{L}(e^{at}f)(z) = \mathbf{L}(f)(z-a).$$

The effect of weighting the Laplace transform of f by $\exp(az)$ depends on the sign of the real number a. More precisely, let a be any non–negative real number. Using the above complex inversion formula, one has:

$$\mathbf{L}^{-1}\big(e^{az}\mathbf{L}(f)(z)\big)(t) = f(t+a),$$

for any $t>0$. Weighting with $\exp(-az)$ gives the following *shifting theorem*:

$$\mathbf{L}^{-1}\big(e^{-az}\mathbf{L}(f)(z)\big)(t) = \mathbf{1}_{(a,\infty)}(t)f(t-a),$$

for any positive real number t. Herein $\mathbf{1}_X$ is the characteristic function of any set X. This is proved calculating the Laplace transform of the right hand side.

The *integration and differentiation principles* for the Laplace transform describe the behaviour of the Laplace transform upon multiplication by integral powers of the complex variable z. Using partial integration in the defining integral of the Laplace transform: $\mathbf{L}(f^{(n)})(z) = z^n \cdot \mathbf{L}(f)(z) - f^{(0)}(0)z^{n-1} - \cdots - f^{(n-1)}(0)z^0$, for any non–negative integer n and any complex number z in the half–plane of convergence for $\mathbf{L}(f)$. In particular,

$$\mathbf{L}\big(f^{(n)}\big)(z) = z^n \cdot \mathbf{L}(f)(z),$$

if f is such that for any non–negative integer k less than n its k–th derivative $f^{(k)}$ vanishes at zero. This last formula holds in general if one considers n–fold integration as differentiation of the negative order $-n$. More precisely, the Laplace inverse of $z^{-n} \cdot \mathbf{L}(f)(z)$, for any non–negative integer n, is given as the n–fold *iterated integral $I_n(f)$* of f. This is recursively defined by $I_0(f) = f$, and

$$I_{m+1}(f)(t) = \int_0^t I_m(f)(u)\,du\,,$$

for any positive real number t, and any non–negative integer m.

Part I   Paris options

4. Black–Scholes framework for valuating contingent claims: For our analysis we place ourselves in the Black–Scholes framework and use the risk–neutral approach to valuating contingent claims. In this set–up one has two securities. First there is a riskless security, a bond, that has the continuously compounding positive interest rate r. Then there is a risky security whose price process S is modelled as follows. Start with a complete probability space equipped with the standard filtration of a standard Brownian motion on it that has the time set $[0, \infty)$. On this filtered space one has the *risk neutral measure Q*, a probability measure equivalent to the given one, and a standard Q–Brownian motion B such that S is the strong solution of the following stochastic differential equation:

$$dS_t = (r - \delta) \cdot S_t \cdot dt + \sigma \cdot S_t \cdot dB_t, \qquad t \in [0, \infty).$$

Herein the positive constant σ is the volatility of S. The constant δ depends on the security modelled. For instance, it is zero if S is a non–dividend paying stock.

5. Paris options: This paragraph dicusses the Paris barrier options of [**CJY**, §2]. Their new idea for cushioning the impact of the underlying hitting the barrier is as follows. They require the underlying S to spend a minimum time $D > 0$ above or below their prespecified barrier $L \geq 0$ before the option is knocked in or knocked out.

The *Paris down–and–in call* to be considered in the sequel is the following European–style contingent claim on S written at time t_0 and with time to maturity T. Its payout at T is that of a call on S with exercise price K:

$$(S_T - K)^+ = \max\big\{S_T - K, 0\big\},$$

if S_t is less than L during a connected subperiod of length at least D of the monitoring period $[t_0, T]$. Equivalently, there is a point in time a such that the interval $I = (a, a + D)$

is contained in the lifespan $[t_0, T]$ of the option and $S_t < L$ for any t in I. Otherwise the call expires worthless.

In the sequel, fix any time t during $[t_0, T]$ at which the Paris down–and–in call has not yet been knocked in. The time–t value $C_{d,i}$ of this call is then a function of K, $\tau = T - t$, S_0, r, δ, L and D. In the sequel, mostly any of these arguments is suppressed from the notation. The valuation problem depends on an excursion time $H_{L,t} = H_{L,D,t}^-$ defined in the sequel such that the Paris down–and–in call is knocked in before its maturity T if and only if $H_{L,t} \leq T$. Using the arbitrage pricing principle, the time–t price $C_{d,i}$ of the Paris down–and–in call is then given by the conditional Q–expectation:

$$C_{d,i} = e^{-r\tau} E_t \Big[\phi(S_T) \cdot \mathbf{1}_{\{H_{L,t} \leq T\}} \Big],$$

with $\phi(S_T)$ being equal to the plain vanilla call payout $(S_T - K)^+$, and denoting the characteristic function of the event $H_{L,t} \leq T$ by $\mathbf{1}_{\{H_{L,t} \leq T\}}$.

In defining $H_{L,t}$ at time t two constellations are to be distinguished. Namely, either to-day's security price S_t is equal to or above the barrier L, or it is below L. In the first case with $S_t \geq L$, the Paris down–and–in call is knocked in before its maturity T iff S remains smaller than L for all points in time of an interval of length at least D contained in $[t, T]$. Thus let $H_{L,t}$ in this case denote the first time s greater than or equal to $t + D$ such that $S_u < L$ for any time u in the interval $(s - D, s)$.

In the second case where $S_t < L$, the security price S has already been staying below L for some connected period time during the lifetime of the option. The down–and–in call is then knocked in before its time to maturity T if the following happens. The security price S continues to stay below L also during the period of time from today until the point in time $t + d$ with d smaller than D, i.e., one has $S_u < L$ for all u in $(t, t + d)$. Thus define $H_{L,t} = t + d$ in this case. If the level L is hit by S before time $t + d$, however, the clock for the minimum lenght D is restarted. So define $H_{L,t}$ in this case to be the first time s greater than or equal to $t + D$ such that $S_u < L$ for any u in the subinterval $(s - D, s)$ of the positive real line.

The discussion of Paris barrier options would not be complete without having referred to [**CJY**, §2] for details on the whole family of Paris barrier options. Calls and puts with either of the following barrier types: down–and–out, down–and–in, up–and–out, up–and–in. Valuating these eight types of options is reduced in a standard way to the case of the above Paris down–and–in call. Indeed, put–call parities are given in [**CJY**, §5], and [**CJY**, §4.2] reduces the valuation of the out–calls to that of the in–calls, mutatis mutandis. This paper thus concentrates on valuating the Paris down–and–in call, in the sequel also referred to as *the Paris option* for simplicity.

Part II   Laplace transforms

6. Laplace transforms of the value of the Paris option: This paragraph discusses the Laplace transforms of the densities for the the Paris option of paragraph five.

The basic valuation identity: This notion of density is made precise by by the following *basic valuation identity* for the time–t price $C_{d,i}$ of the Paris option:

$$C_{d,i} = e^{-\left(r + \frac{\varpi^2}{2}\right) \cdot \tau} \int_{\beta(S_t)}^{\infty} e^{\varpi x} \cdot \left(S_t \cdot e^{\sigma x} - K\right) \cdot h_b(\tau, x) \, dx \,,$$

with the *Paris option density* h_b given by:

$$h_b(u, y) = \int_{-\infty}^{\infty} E^* \left[\mathbf{1}_{\{H_b^* < u\}} \frac{1}{\sqrt{2\pi \cdot (u - H_b^*)}} \cdot e^{-\frac{(x-y)^2}{2 \cdot (u - H_b^*)}} \right] d\mu^*(x) \,,$$

for any real numbers $u > 0$ and y, and where $\beta(S_t) = \sigma^{-1} \log(K/S_t)$, where $H_b^* = H_{L,t} - t$, and with $\mu^* = \mu_{b,D,-}$ the measure for Brownian motion at time H_b^*. By construction, $h_b(u, \ )$ is zero for u less than D, respectively d depending on today's situation.

Laplace transforms of the densities: With the proof of the basic valuation identity postponed to the end of this paragraph, the Laplace transforms of h_b with respect to time are discussed next. For any fixed real number y they are recalled to be defined by:

$$\mathbf{L}\big(h_b(\ , y)\big)(z) = \int_0^{\infty} e^{-zt} h_b(t, y) \, dt \,,$$

for any complex number z with sufficiently big real part. Moreover define the function Ψ by:

$$\Psi(z) = \int_0^{\infty} x \cdot e^{-\frac{x^2}{2} + zx} \, dx \,,$$

for any complex number z, and choose on the complex plane with the non–positive real axis deleted the square root defined using the principal branch of the logarithm. The computations of [**CJY**, §§5, 8] can then be summarized as follows.

Proposition A: *For any real number y, the Laplace transform of $h_b(\ , y)$ is a holomorphic function on the right complex half–plane.*

Proposition B: *Suppose $b = \sigma^{-1} \log(L/S_t)$ is non–positive. For any fixed real number y, the Laplace transform of $h_b(\ , y)$ is given by:*

$$\mathbf{L}\big(h_b(\ , y)\big)(z) = \frac{e^{\frac{b}{\sqrt{D}} \sqrt{2Dz}}}{\sqrt{D} \sqrt{2Dz} \, \Psi(\sqrt{2Dz})} \int_0^{\infty} x \cdot e^{-\frac{x^2}{2D} - |b - x - y| \sqrt{2z}} \, dx \,,$$

for any complex number z with positive real part.

If $b = \sigma^{-1} \log(L/S_t)$ is positive, i.e., when today's security price S_t is below the barrier L, we have modified the Paris option of [**CJY**]. Today's situation is then such that the price of the security has already been staying below the barrier L for a certain connected period of time during the lifetime of the Paris options. For the Paris option to be knocked in the price thus has to stay below the barrier L only for another connected period time of a length $d < D$ from today, i.e., $S_u < L$ for all u in $[t, t+d)$. In this way, Paris options now can be valuated at any point of their monitoring periods. The desirability of this modification has been pointed out to me by Pliska. I regard the following modification of the valuation results of [**CJY**] as a direct result of his suggestions.

Proposition C: *Suppose* $b = \sigma^{-1} \log(L/S_t)$ *is negative. For any real number* y, *the Laplace transform of* $h_b(\ ,y)$ *is given on the right half–plane as the following four–term sum of Laplace transforms:*

$$\mathbf{L}\big(h_b(\ ,y)\big) = \mathrm{Erfc}\left(\frac{b}{\sqrt{2d}}\right) \cdot \mathbf{L}\left(h_{b,1}(\ ,y)\right) + \mathbf{L}\left(h_{b,2}(\ ,y)\right)$$

$$+ \mathrm{Erfc}\left(\frac{b}{\sqrt{2d}}\right) \mathrm{Erf}\left(\frac{b}{\sqrt{2d}}\right) \cdot \mathbf{L}\left(h_{b,3}f(\ ,y)\right) + \mathrm{Erf}\left(\frac{b}{\sqrt{2d}}\right) \cdot \mathbf{L}\left(h_{b,4}(\ ,y)\right)$$

Herein $h_{b,k}$ are the functions on the positive real line times the real line defined by:

$$h_{b,3}(u,y) = \mathbf{1}_{(d,\infty)}(u) \cdot \frac{1}{u-d+D}\left\{ \frac{\sqrt{u-d}}{\sqrt{2\pi}} \cdot e^{-\frac{(y-b)^2}{2(u-d)}} \right.$$

$$\left. -\frac{1}{2}\frac{(y-b)\sqrt{D}}{\sqrt{u-d+D}} \cdot e^{-\frac{(y-b)^2}{2(u-d+D)}} \mathrm{Erfc}\left(\frac{(y-b)\sqrt{D}}{\sqrt{2(u-d)(u-d+D)}}\right) \right\},$$

$$h_{b,4}(u,y) = \mathbf{1}_{(d,\infty)}(u)\frac{1}{\sqrt{2\pi u}} \cdot f_{b,u}(y) \qquad \text{where} \qquad f_{b,u}(y) = e^{-\frac{y^2}{2u}} - e^{-\frac{(y-2b)^2}{2u}},$$

for any positive real number u and any real number y, while the remaining two functions are defined using their Laplace transforms as follows:

$$\mathbf{L}\big((h_{b,1}(\ ,y))\big)(z) = \int_0^d \frac{e^{-zw}}{D\sqrt{2z}\,\Psi(\sqrt{2Dz})}\left(\int_0^\infty x \cdot e^{-\frac{x^2}{2D} - |b-x-y|\sqrt{2z}} dx\right)\mu_b(dw),$$

$$\mathbf{L}\big((h_{b,2}(\ ,y))\big)(z) = \frac{1}{\sqrt{2\pi d}}\int_0^d \frac{e^{-zw}}{\sqrt{2z}\,\Psi(\sqrt{2Dz})}\left(\int_{\mathbf{R}} e^{-|x-y|\sqrt{2z}} f_{b,d}(x)dx\right)\mu_b(dw),$$

for any positive real number y and any complex number z with positive real part. Herein μ_b denotes the law of the first passage time to the level b, given by

$$\mu_b(dw) = \frac{b}{\sqrt{2\pi}}w^{-3/2}e^{-\frac{b^2}{2w}},$$

on the positive real line.

Proof of the basic valuation identity: The proof of the basic valuation identity is by reduction to that of [**CJY**, §§4, 5]. As a first step, the valuation problem for the Paris option of paragraph two is transcribed in in terms of a restarted–at–time–t normalized Brownian motion with drift W^* as follows. With B^* the restarted–at–time–t Brownian motion $B^*(u) = B(t+u) - B(t)$, consider the Brownian motion with drift:

$$W_u^* = \frac{1}{\sigma}\log\left(\frac{S_{t+u}}{S_t}\right) = \varpi u + B_u^*, \qquad u \in [t, \infty),$$

whose drift coefficient is $\varpi = \sigma^{-1} \cdot (r - \delta - \sigma^2/2)$. Recalling $b = \sigma^{-1} \cdot \log(L/S_t)$, one has $S_{t+u} < L$ iff $W^*(u) < b$ by construction. Thus define the excursion time $H_b^* = H_{b,D,0}^{*,-}$ for W^* as follows. If today's value $W^*(0)$ of W^* is greater than or equal to b then H_b^* is the first time s greater than or equal to D such that $W^*(u) < b$ for any time u in the interval $(s-D, s)$ of the positive real line. Suppose today's value $W^*(0)$ of W^* is less than b. Set $H_b^* = d$ if $W^*(u) < b$ for any time u in the interval $[0, d)$ remaining for W^* to have stayed a connected period of time of length D below L. Otherwise, H_b^* is the first time s greater than or equal to D such that $W^*(u) < b$ for any time u in the subinterval $(s-D, s)$ of the positive real line. One has $H_b^* = H_{L,t} - t$, whence $H_{L,t} \leq T$ iff $H_b^* \leq \tau = T - t$. Using Strong Markov, the valuation problem for the Paris option thus transcribes as follows:

$$C_{d,i} = e^{-r\tau} E^Q \left[\left(S_t \cdot e^{\sigma W^*(\tau)} - K \right)^+ \cdot \mathbf{1}_{\{H_b^* \leq \tau\}} \right].$$

As a next step, using the particular case [**KS**, p.196f] of the Cameron–Martin–Girsanov theorem, change measure from Q to Q^* such that the drift term in W^* is killed and W^* becomes a Q^*–Brownian motion. Change measure in the expectation expressing the price $C_{d,i}$ and iterate the Q^*–expectation thus obtained to get:

$$C_{d,i} = e^{-\left(r + \frac{\varpi^2}{2}\right) \cdot \tau} E^* \left[\mathbf{1}_{\{H_b^* \leq \tau\}} E^* \left[e^{\varpi W_\tau^*} \left(S_t \cdot e^{\sigma W_\tau^*} - K \right)^+ \Big| \mathbf{F}_{H_b^*} \right] \right].$$

Using H_b^* as $\mathbf{F}$–stopping time apply the strong Markov property of Brownian motion in the conditional expectation. The resolvent thus obtained can be further simplified with the random variables H_b^* and $W^*(H_b^*)$ being independent. With μ^* the measure for $W^*(H_b^*)$, the basic valuation identity follows.

7. An intermediate identity for the proofs of the Laplace transforms: Adopt the notation introduced in the previous paragraph six in the basic valuation identity for Paris options. Then one has the following intermediate identity:

Lemma: *For any real number y, one has:*

$$\mathbf{L}\big(h_b(\ ,y)\big)(z) = \int_{\mathbf{R}} E^* \left[e^{-z H_b^*} \right] \cdot \frac{e^{-|x-y|\sqrt{2z}}}{\sqrt{2z}} \, \mu^*(dx),$$

for any complex number z in the right complex half–plane.

For proving the Lemma interchange the Laplace transform with the two exponential integrals defining the Paris option density $h_b(u, y)$ in paragraph six to get:

$$\mathbf{L}\big(h_b(\ ,y)\big)(z) = \int_{\mathbf{R}} E^* \left[\int_{H_b^*}^{\infty} e^{-zu} \frac{1}{\sqrt{2\pi(u - H_b^*)}} \cdot e^{-\frac{(x-y)^2}{2(u - H_b^*)}} \, du \right] \mu^*(dx).$$

In this last integral successively change variables $w = u - H_b^*$ and separate the expectation from the Laplace transform to get:

$$\int_{\mathbf{R}} E^* \left[e^{-z H_b^*} \right] \cdot \mathbf{L}\left(\frac{1}{\sqrt{2\pi u}} \cdot e^{-\frac{(x-y)^2}{2u}} \right)(z) \, \mu^*(dx).$$

Using [**Sch**, §3] for the Laplace transform completes the proof of the Lemma.

8. Computing the Laplace transform in the case b equal to zero: The computation of the Laplace transforms of paragraph six for h_b is by reduction to the case b equal to zero where one has to show the *key relation*:

$$E^*\left[e^{-\frac{z^2}{2}H_0^*}\right]\cdot\Psi(z\sqrt{D})=\Psi(0)=1,$$

for any complex number z with positive real part. This paragraph reviews the key ideas of its proof in [**CJY**, §8]. The computations in this case b equal to zero are based on the Azéma martingale and properties of the Brownian meander. The *Azéma martingale* is the martingale on $[0,\infty)$ for the slow Brownian filtration $\mathbf{F}^+$ given for any $t\geq 0$ by:

$$\mu_t=(\operatorname{sgn}W_t^*)\sqrt{t-g_t}\,.$$

Herein g_t is defined as the supremum over all real numbers $s\leq t$ such that $W^*(s)=0$. The process given by:

$$m_t(u)=\frac{1}{\sqrt{t-g_t}}\left|W^*\big(g_t+u(t-g_t)\big)\right|,\qquad u\in[0,1]$$

is a *Brownian meander* and is independent of the σ–subfield $\mathbf{F}^+(g_t)$ of the slow Brownian filtration. Its law is independent of t and given by $x\exp\left(-x^2/2\right)\cdot\mathbf{1}_{(0,\infty)}\,dx$. The tautology $W^*(t)=m_t(1)\cdot\mu_t$ thus implies the following identity:

$$E^*\left[e^{zW_t^*-\frac{z^2}{2}t}\,\middle|\,\mathbf{F}^+(g_t)\right]=e^{-\frac{z^2}{2}t}\cdot\Psi(z\cdot\mu_t)\,,$$

for any positive real number z. Fixing any such z, the crucial point is to exhibit the product $\psi(-z\cdot\mu(H_0^*))\exp\left(-z^2H_0^*/2\right)$ as a martingale. This is achieved by a boundedness argument showing the argument of the above conditional expectation to be uniformly integrable for any time t up to H_0^*. Using another form of the optional stopping theorem it follows that the Q^*–expectation of the martingale stopped at H_0^* equals the Q^*–expectation of it at time zero, whence $\Psi(0)$ which equals one. With the random variables H_0 and $W^*(H_0^*)$ independent, the key relation results. This is an identity between functions holomorphic in particular on the right half–plane. Using the identity theorem, it thus remains valid for any complex number z with positive real part.

9. Computing the Laplace transform for b non–positive: Using §7 Lemma the proof of §6 Proposition B reduces to compute the expectation of $\exp\left(-zH_b^*\right)$, for any complex number z with positive real part, and the density μ^*. Following [**CJY**, §8.3.3], this is by reduction to the case b equal to zero of the previous paragraph. Indeed, decompose H_b^* as follows:

$$H_b^*=T_b+H_0^{**}.$$

Herein T_b is the first passage time of W^* to the level b, and H_0^{**} is defined as follows. It is the smallest non–negative point in time s at which the restarted–at–time–T_b Brownian motion $W^{**}(u):=W^*(T_b+u)-W^*(T_b)=W^*(T_b+u)-b$ is zero for the first time after having been less than zero for a connected period of time of length at least D.

To compute the expectation required, notice that T_b and H_0^{**} are independent random variables. The conditional expectation at time T_b of $\exp\left(-(w^2/2)H_b^*\right)$ thus is the product

of $\exp\left(-(w^2/2)T_b\right)$ and the expectation of $\exp\left(-(w^2/2)H_0^{**}\right)$. With the law of H_0^{**} equivalent to that of H_0^*, this last expectation is given by the key relation of paragraph eight. So it is deterministic in particular. Take expectations to time zero of this product. The expectation of $\exp\left(-(w^2/2)T_b\right)$ that remains to be computed is the Laplace transform of the law of T_b at $w^2/2$. Using [**Sch**, §3] it is equal to $\exp\left(-|b|(2 \cdot (w^2/2))^{1/2}\right)$ and thus to $\exp\left(+bw\right)$. With $w = (2z)^{1/2}$ one gets:

$$E^*\left[e^{-z \cdot H_b^*}\right] = \frac{1}{\Psi(\sqrt{2Dz}\,)} \cdot E^*\left[e^{-\sqrt{2z} \cdot T_b}\right] = \frac{1}{\Psi(\sqrt{2Dz}\,)} \cdot e^{b\sqrt{2z}} \ .$$

To determine μ^* notice the tautology $W^*(H_b^*) = W^{**}(H_0^{**}) + b$. Thus $W^*(H_b^*) \le x$ iff $W^{**}(H_0^{**}) \le x - b$. From the discussion of the case b equal to zero, the law of $W^{**}(H_0^{**})$ is that of the negative of m_1. As a consequence:

$$Q^*\left(W^*(H_b^*) \in dx\right) = \mathbf{1}_{(-\infty, b]}(x) \cdot (b - x) \cdot e^{-\frac{(x-b)^2}{2D}} \cdot \frac{dx}{D} \ .$$

Substitute into §7 Lemma and change variables $w = b - x$ to obtain §6 Proposition B for any complex number z with sufficiently big positive real part. Extend to the whole right half–plane using the consequence of the key relation of paragraph eight that $\Psi((2Dz)^{1/2})$ has no zeroes there. This completes the proof of §6 Proposition B and the b–non–positive part of §6 Proposition A.

10.   Computing the Laplace transform for b positive: The proof of Proposition C of paragraph six modifies the argument of [**CJY**, §8.3.3]. Using §7 Lemma, the first task is to compute the density μ^* and the Q^*–expectation of $\exp\left(-zH_b^*\right)$ for any complex number z with positive real part. The two cases where the level b is hit before or after time d are to be distinguished. Thus decompose the underlying probablity space Ω into the set A on which the first passage time T_b of W^* to the level b is less than or equal to d and its complement $\Omega \setminus A$ on which T_b is bigger than d. This induces the decomposition:

$$E^*\left[e^{-zH_b^*}\right] = E^*\left[\mathbf{1}_A \cdot e^{-zH_b^*}\right] + E^*\left[\mathbf{1}_{\Omega \setminus A} \cdot e^{-zH_b^*}\right]$$

of the Q^*–expectation of the random variable $\exp\left(-zH_b^*\right)$. By construction $H_b^* = d$ on the complement of A. So the second above summand is $\exp\left(-zd\right)Q^*(T_b > d)$. On the set A the level b is reached before the critical time d and the clock for the excursion is reset to zero at the first passage time T_b to the level b. Accordingly one has the decomposition:

$$H_b^* = T_b + H_0^{**},$$

with H_0^{**} defined as follows. It is the smallest $s \ge 0$ at which the restarted–at–time–T_b Brownian motion $W^{**}(u) := W^*(T_b + u) - W^*(T_b) = W^*(T_b + u) - b$ is zero for the first time after having been less than zero for a connected period of time of lenght at least D.

With this random variable H_0^{**} independent from T_b, the conditional expectation of the random variable $\mathbf{1}_A \cdot \exp\left(-zH_b^*\right)$ at time T_b thus equals $\exp\left(-zT_b\right)$ times the expectation of $\exp\left(-zH_0^{**}\right)$. The key relation of paragraph eight now applies to H_0^{**} and identifies this last expectation as the reciprocal of $\Psi((2zD)^{1/2})$. The expectation of $\mathbf{1}_A \cdot \exp\left(-zT_b\right)$

on the other hand, is given by integrating $\exp{(-zw)}$ from zero to d against the density μ_b of T_b. Summarizing, it so follows for any complex number z with positive real part:

$$E^*\left[e^{-zH_b^*}\right] = Q^*(T_b > d)e^{-zd} + \frac{\int_0^d e^{-zw}\,\mu_b(dw)}{\Psi(\sqrt{2Dz}\,)}\;.$$

For determining μ^* decompose the random variable $W^*(H_b^*)$ with respect to A:

$$W^*(H_b^*) = \left(W^* * (H_0^{**}) + b\right) \cdot \mathbf{1}_A + W^*(d) \cdot \mathbf{1}_{\Omega\setminus A}\;.$$

Using the independence of T_b and H_0^{**}, the law of the first summand is obtained as the convolution of the laws of T_b and $W^{**}(H_0^{**})$, whence

$$\mu_1^*(dx) = Q^*\left((W^{**}(H_0^{**}) + b) \in dx; T_b \le d\right)$$

$$= Q^*(T_b \le d) \cdot \mathbf{1}_{(-\infty,b]}(x) \cdot (b-x) \cdot e^{-\frac{(x-b)^2}{2D}} \cdot \frac{dx}{D}\;.$$

For the law of the second summand notice $T_b > d$ iff $W^*(t) < b$ for all $t \le d$, or equivalently, $\max\{W^*(t) | t \le d\}$ is smaller than b. Using [**H**, p.9] one so has:

$$\mu_2^*(dx) = Q^*\left(W^*(d) \in dx; T_b > d\right)$$

$$= e^{-\frac{x^2}{2d}} \cdot \frac{dx}{\sqrt{2\pi d}} - e^{-\frac{(x-b)^2}{2D}}\,dx \cdot \frac{dx}{\sqrt{2\pi d}}\;.$$

With the expectation factor in §7 Lemma seen above to be deterministic and independent of the variable x, the Laplace transform of $h_b(\ ,y)$ at z is given as the product:

$$\mathbf{L}\left(h_b(\ ,y)\right)(z) = E^*\left[e^{-zH_b^*}\right]\int_{\mathbf{R}} \frac{e^{-|x-y|\sqrt{2z}}}{\sqrt{2z}}\,(\mu_1^* + \mu_2^*)(dx)\;.$$

One is so reduced to compute the two integrals of the second factor of this product. For the first of these, on substituting for μ_1^* and changing variables $w = b-x$, one gets:

$$\int_{\mathbf{R}} \frac{e^{-|x-y|\sqrt{2z}}}{\sqrt{2z}}\,\mu_1^*(dx) = \frac{Q^*(T_b \le d)}{D}\int_0^\infty x \cdot \frac{e^{-|b-x-y|\sqrt{2z}}}{\sqrt{2z}} \cdot e^{-\frac{x^2}{2D}}\,dx\;.$$

For the second of these one analogously obtains:

$$\int_{\mathbf{R}} \frac{e^{-|x-y|\sqrt{2z}}}{\sqrt{2z}}\,\mu_2^*(dx) = \frac{1}{\sqrt{2\pi d}}\int_{\mathbf{R}} \frac{e^{-|x-y|\sqrt{2z}}}{\sqrt{2z}}\left(e^{-\frac{x^2}{2d}} - e^{-\frac{(x-2b)^2}{2d}}\right)dx\;.$$

At this point it remains to identify the two Laplace inverses of §6 Proposition C to complete its proof. This is based on the result:

$$\mathbf{L}^{-1}\left(\frac{1}{\sqrt{2z}} \cdot e^{-\alpha\sqrt{2z}}\right)(u) = \frac{1}{\sqrt{2\pi u}} \cdot e^{-\frac{\alpha^2}{2u}}$$

valid for any positive real numbers α and u, and discussed in [**Sch**, §3]. Applying Fubini's theorem to get the inversion integral as inner integral, the improper integral factor of the above μ_1^* integral becomes:

$$\int_0^\infty x e^{-\frac{x^2}{2D}} \cdot \mathbf{L}^{-1}\left(e^{-zd}\frac{e^{-|b-x-y|\sqrt{2z}}}{\sqrt{2z}}\right)(u)\, dx \, .$$

Successively apply the shifting theorem for the Laplace transform of paragraph three to take care of the factor $\exp(-zd)$ and the above Laplace inversion formula with α equal to $|b-x-y|$ to arrive at:

$$\frac{1}{D} \cdot \mathbf{1}_{(d,\infty)}(u)\frac{1}{\sqrt{2\pi(u-d)}} \int_0^\infty x \cdot e^{-\left(\frac{x^2}{2D}+\frac{(b-x-y)^2}{2(u-d)}\right)} dx \, .$$

This integral is seen to be the value of the function $h_{b,3}$ at u and y. Analogously, the Laplace inverse at any positive real number u of the integral factor of the above μ_2^* integral is:

$$\frac{1}{\sqrt{2\pi d}}\mathbf{1}_{(d,\infty)}(u)\frac{1}{\sqrt{2\pi(u-d)}} \int_\mathbf{R} e^{-\frac{(x-y)^2}{2(u-d)}}\left(e^{-\frac{x^2}{2d}} - e^{-\frac{(x-2b)^2}{2d}}\right)dx$$

and is seen to be equal to the value of the function $h_{b,4}$ at u and y. This completes the proof of §6 Proposition C.

Part III   Analytic properties of the function Ψ

11.   Analytic properties of Ψ: This paragraph collects and discusses pertinent analytical properties of the function Ψ for any complex number w recalled to be given by:

$$\Psi(w) = \int_0^\infty x \cdot e^{-\frac{x^2}{2}+wx}\, dx \, .$$

Developing the linear exponential factor of the integrand in its series and integrating the resulting series term by term, gives the following *series expansion*:

$$\Psi(w) = \sum_{n=0}^\infty \frac{2^{\frac{n}{2}}}{n!} \cdot \Gamma\left(\frac{n+2}{2}\right) \cdot w^n \, .$$

This series is absolutely convergent for any complex number w, and convergence is uniform on compact sets. Rearrange it in its even and odd order terms. Replacing w by its negative leaves unchanged the even part and produces a minus sign in the odd part. Using the duplication identity for the gamma function and redeveloping the square order exponential series that results as a factor, then gives the following *key identity*:

$$\Psi(w) = \Psi(-w) + \sqrt{2\pi} \cdot w \cdot e^{\frac{w^2}{2}}$$

connecting the values of Ψ on the right half–plane with those on the left half–plane.

12.   A uniform asymptotic expansion: The aim of this paragraph is to prove for the function Ψ given by $\Psi(w) = \int_0^\infty x \exp\left(-x^2/2 + wx\right) dx$, for any complex number w, the following uniform asymptotic expansion.

Lemma: *For any complex number w with positive real part, one has the expansion:*

$$\Psi(-w) = \frac{1}{\sqrt{\pi}} \sum_{k=1}^N (-1)^{k+1} \cdot 2^k \cdot \Gamma\left(k + \frac{1}{2}\right) \cdot \frac{1}{w^{2k}} + R_{N+1}(w),$$

for any positive integer N, whose remainder term R_{N+1} satisfies the estimate:

$$\left|R_{N+1}(w)\right| \le 2 \cdot \frac{(2N+1)!}{N!} \cdot \frac{1}{|w|^{2(N+1)}}.$$

Corollary: *For any complex number z not a non–positive real number, one has:*

$$\Psi(-\sqrt{z}) = \frac{2}{z} + R_2(\sqrt{z})$$

with $R_2(z^{1/2})$ being big Oh in $|z|^{-2}$ for the absolute value of z going to infinity.

The Corollary is an immediate consequence of the Lemma. Recall that the principal branch of the logarithm has been chosen. The proof of the Lemma is in two steps. As a first step, establish the asymptotic expansion with the remainder term satisfiying the estimate:

$$\left|R_{N+1}(w)\right| \le \frac{C_N}{\cos^{2(N+1)}(\pi/2 - \delta_z)} \cdot \frac{1}{|w|^{2(N+1)}} \qquad \text{where} \qquad C_N = \frac{(2N+1)!}{N! \cdot 2^N},$$

if $|\arg(w)| \le \pi/2 - \delta_w$ for $\delta_w > 0$. For this develop the square exponential factor of the integrand for $\Psi(-w)$ into its Taylor series up to the $(N-1)$–st term and integrate the resulting sum term by term. Using the duplication formula for the gamma function the coefficients of the asymptotic expansion follow on shifting the summation index by one. Proceeding analogously, the absolute value of the remainder term can be majorized by C_N times the reciprocal of the $2(N+1)$–st power of the real part of w. With w in the right half–plane, $\mathrm{Re}(w)$ equals $|w| \cdot \cos(\theta)$ for an angle θ of absolute value less than or equal to $\pi/2 - \delta_w$ as above. Thus $\cos(\theta)$ is bigger than the positive constant $\cos(\pi/2 - \delta_w)$ completing the proof of the first step.

As a second step, extend the above asymptotic expansion by showing the above estimate to be valid also for any complex numbers w such that $\pi/4 < |\arg(w)| \le 3\pi/4 - \delta_w$. This is based on the following integral representation:

$$\Psi(-w) = \int_0^{e^{i\theta} \cdot \infty} \xi \cdot e^{-\frac{\xi^2}{2} - w\xi} \, d\xi,$$

for any complex number w. Herein θ is any fixed any angle of absolute value less than $\pi/4$ and the integral is over the path sending any positive real number R to $R \cdot \exp(i\theta)$. This representation is proved using Cauchy's theorem. Indeed, the absolute value of the above integrand is majorized by $|\xi| \cdot \exp\left(-\mathrm{Re}(\xi^2)/2 - \mathrm{Re}(w\xi)\right)$. It thus converges uniformly to

zero with the radius R going to infinity on that part of the circle in the origin of radius R which is parametrized by angles of absolute value less than $\pi/4$.

Now fix any positive real number ε smaller than δ_w. If $\arg(w)$ is negative, choose the angle $\theta = +\pi/4-\varepsilon$ in the above integral representation. Otherwise let $\theta = -\pi/4+\varepsilon$. In the sequel the case $\arg(w)$ positive is considered. In the above integral representation then develop the square exponential factor of the integrand into its Taylor series up to the $(N-1)$–st order. In the identity:

$$\int_0^{e^{i\theta}\cdot\infty} \xi^{2k+1} e^{-w\xi}\, d\xi = e^{2(k+1)i\theta} \int_0^\infty R^{2k+1} e^{-we^{i\theta}R}\, dR$$

write $w\exp(i\theta) = |w|\exp(i(\theta+\arg(w)))$. Notice that the angle $\theta+\arg(w))$ is between $\theta+\pi/4 = \varepsilon$ and $\theta+3\pi/4-\delta_w = \pi/2-(\delta_w-\varepsilon)$. Thus it is smaller than $\pi/2$ in particular. As a consequence, the real part of $\eta = w\exp(i\theta)$ is positive. Recalling that $\Gamma(a+1)$ is equal to the integral $\eta^{a+1}\int_0^\infty \exp(-\eta t)t^a\, dt$, the above integral is thus seen to be equal to $w^{-2(k+1)}\cdot\Gamma(2(k+1))$. The coefficients of the asymptotic expansion follow. Proceeding analogously, the absolute value of the remainder term $R_{N+1}(w)$ is majorized by C_N times the reciprocal of the $2(N+1)$–st power of $|w|\cdot\cos(\theta+\arg(z))$. Recalling the above bounds for its argument, the cosine herein is positive and bounded from below by $\cos(\pi/2-(\delta_w-\varepsilon))$, for any positive ε less than δ_w. Thus it is bounded from below by $\cos(\pi/2-\delta_w)$, completing the proof of the second step.

To complete the proof of the uniform asymptotic expansion, apply the estimate of the first step with any δ_z strictly between $\pi/4$ and $\pi/2$ if $|\arg(w)|$ is less than $\pi/4$. Otherwise, apply the estimate of the second step with δ_w strictly between $\pi/4$ and $\pi/2$. In both cases, $\cos(\pi/2-\delta_w)$ is minorized by $\cos(\pi/4) = 2^{-1/2}$. Cancelling the powers of two one gets the factor two. This completes the proof of the uniform asymptotic expansion.

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On the valuation of Paris barrier options: foundational results

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